

Higher-Order Interactions: Understanding the knowledge capacity of social groups using simplicial sets

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Abstract A predominant benefit of social living is the ability to share knowledge that cannot be gained through the information an individual accumulates based on its personal experience alone. Traditional computational models have portrayed sharing knowledge through interactions among members of social groups via dyadic networks. Such models aim at understanding the percolation of information among individuals and groups to identify potential limitations to successful knowledge transfer. However, because many real-world interactions are not solely pairwise, i.e., several group members may obtain information from one another simultaneously, it is necessary to understand more than dyadic communication and learning processes to capture their full complexity. We detail a modeling framework based on the simplicial set, a concept from algebraic topology, which allows elegant encapsulation of multi-agent interactions. Such a model system allows us to analyze how individual information within groups accumulates as the group's collective set of knowledge, which may be different than the simple union of individually contained information. Furthermore, the simplicial modeling approach we propose allows us to investigate how information accumulates via sub-group interactions, offering insight into complex aspects of multi-way communication systems. The fundamental change in modeling strategy we offer here allows us to move from portraying knowledge as a “token”, passed from signaler to receiver, to portraying knowledge as a set of accumulating building blocks from which novel ideas can emerge. We provide an explanation of relevant mathematical concepts in a way that promotes accessibility to a general audience [*Current Zoology* 61 (1): 114–127, 2015].

Keywords Collective decision making, Communication, Cultural transmission, Information flow, Simplex, Social learning

1 Background

Network-based methods have yielded great success elucidating many questions about group structure and function based on pairwise interactions. Efforts to capture interactions between (and implicitly, though incompletely, among) individuals in groups have thus far relied primarily on network representations, in which each individual in the population is represented by a vertex, and interactions between two vertices are represented by an edge. These types of models reveal a structure to the set of interactions which can uncover functions of systems such as food webs (Dunne, 2002; Elton, 1936; Lewis et al., 2002; Memmott et al., 2004; Montoya, 2002; Paine, 1966; Pimm and Lawton, 1977; Sole and Montoya, 2001; Williams, 2002) and mutualistic interaction networks in ecology (Bascompte et al., 2003, Bascompte et al., 2006, Ings et al., 2009; Proulx et al., 2005; Stanko, 2011), and more generally for examining decentralized decision-making in multi-agent systems

(Bode et al., 2011; Gordon, 2002; Olfati-Saber et al., 2007). Elucidation of the structure of these interaction networks allows rigorous examination of various behavioral processes (reviewed in Pinter-Wollman et al., 2014). Some examples include organizational features such as rankings and orderings e.g. social hierarchies (de Silva et al., 2011; Hobson et al., 2015; Dey et al., 2015), consumer/producer relationships (Berlow et al., 2009); interactions in aggression (Beisner et al., 2015), mating (Inghilesi et al., 2015), cooperation (Franz et al., 2015), or communication (Crofoot et al., 2011); and percolation such as energy flow in food networks (Thompson et al., 2005) or opinion adoption in social insects (Fewell, 2003). Furthermore, we can examine how local changes to interactions affect global outcomes i.e., the effects of selfish individual behaviors on the organizational efficiency of social populations (Hock et al., 2010). Net-works are well suited to examine these phenomena because they provide a rigorous framework and structure to the aggregate of pairwise relationships

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between individuals and provide a method of comparison between seemingly unrelated systems (Milo et al., 2002).

Many important properties of social groups exceed the union of dyadic interactions and cannot be efficiently captured using basic network models. Particularly, multi-way interactions and identification of functional sub-groups are constrained using traditional network analysis methods. For example, if four individuals communicate with one another, the network representation of this group scenario is the complete graph on four vertices (Fig. 1B). However, this network depiction cannot express whether the communication that took place was, in fact, among all four individuals as a group, depicted in the 3-dimensional object indicating simultaneous interaction in Fig. 1A, or whether they occurred as six independent pairwise transfers of information, shown in Fig. 1B. Potentially important aspects of the interactions within this group, such as simultaneous information exchange among three members, exceed the union of the pairwise relationships depicted, and would thus exceed the representational and computational methodologies of a simple network. We propose the use of simplicial models which would generalize traditional network approaches by allowing us to encapsulate explicit group interactions of more than two simultaneous participants and, critically, to designate which groups are participating subgroups of others. This tool includes network analysis as the simplest form of simplicial models, but further allows us to exploit the already developed and studied constructs for analysis of higher-

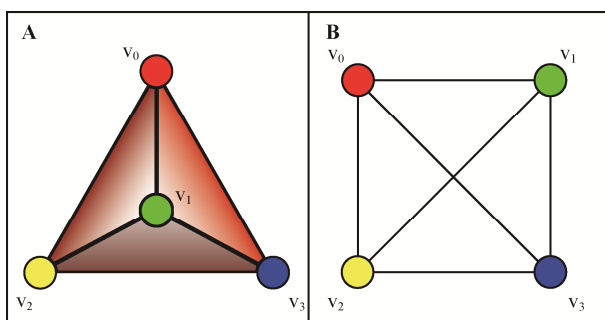


Fig. 1 Simplicial set representations of communication in a group of 4

Simplicial set representations of two of several possible communication scenarios in a group of 4 individuals. **A.** a 3-simplex denotes that all four individuals are participating in a group conversation. **B.** Six different 1-simplices denote that each individual is speaking to every other individual pairwise and privately. Such differentiation between the two types of communication modes is not possible using the traditional network representation, as both scenarios would have a network representation as shown in B.

order groups from homology theory (Friedman, 2008; Goerss and Jardine, 2009; May, 1992; Spivak, 2009). Here we show how simplicial set representations can be used to complement existing network techniques to better describe socially mediated transmission of information. We first introduce simplicial modeling and highlight how it differs from social network analysis. We then provide examples of behavioral processes that can benefit from incorporating this framework followed by a specific application. We conclude the paper by highlighting the insights about animal societies that can be gained by using simplicial modeling.

2 An Introduction to Simplicial Modeling

Simplicial modeling differs from social network analysis in how it represents interactions. Traditional networks depict each individual in the group as a node (or vertex) and the interactions between these individuals as links (or edges) between pairs of vertices. Recent work has advocated the use of the *simplex* as a fundamental representation of interactions between individual entities within a social group making it possible to represent more than dyadic interactions (Kee et al., 2013; Moore et al., 2012; Ramanathan et al., 2011; Spivak, 2009). A simplex can be visualized most easily as an n -dimensional polygon determined by $n + 1$ points (Fig. 2). For example, a triangle together with its interior determined by its three vertices is a 2-dimensional simplex (or 2-simplex). A tetrahedron together with its interior determined by its four vertices is a 3-simplex, and so on. Finally, note that by this definition, an edge in a traditional graph fits the definition of a 1-simplex. The expansion of the traditional network model to higher dimensionality using simplices is intuitive, in that each individual in the group is represented by a 0-simplex (or vertex) and each k -simplex is representative of some interaction that has occurred among the $(k+1)$ vertices that comprise the simplex. If we then “glue” multiple simplices together along particular faces (Fig. 3), we have constructed a *simplicial set* (or more restrictively, a *semi-simplicial set* or a *simplicial complex* (Munkres, 1984; Spanier, 1994)). A formal definition of each term is given in Appendix 1.

Simplicial sets have been well-studied as mathematical constructs in algebraic topology, but little work has been invested in applying them to group interaction dynamics. Simplicial models allow us to represent multi-individual (multi-vertex) encounters as simplices, thereby enabling us to define biologically important pro-

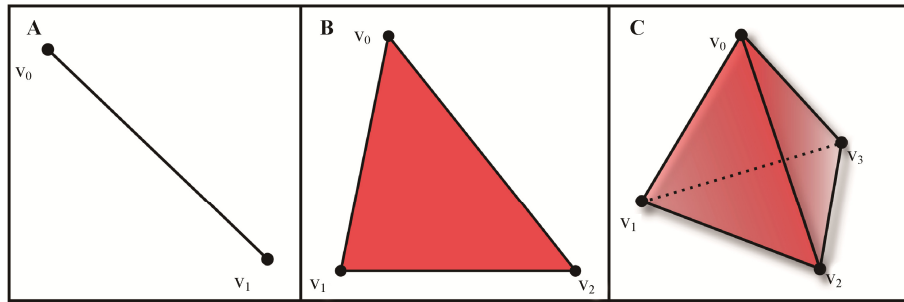


Fig. 2 Geometric representation of a simplex

A simplex is the generalization of a tetrahedron to n dimensions. A k -simplex can be represented geometrically as the polyhedral hull of $(k+1)$ vertices. A 0-simplex is a point or vertex, and simple geometric representations of a 1-, 2-, and 3-simplex are represented here in A, B, and C, respectively.

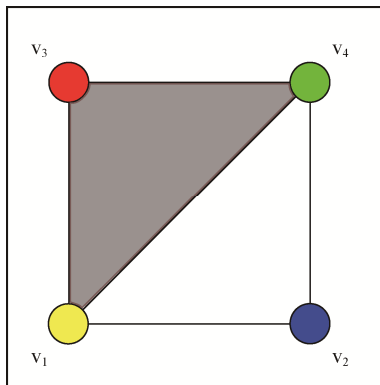


Fig. 3 An example (semi)-simplicial set

An elementary simplicial set constructed from four 0-simplices, five 1-simplices, and one 2-simplex shown by the grey triangle (v_1, v_3, v_4). This topology also satisfies the more restrictive definition of a semi-simplicial set, because it does not contain any degenerate simplices (i.e., simplices with repeated vertices).

cesses, such as learning as a function of group size. Representation of such multi-individual encounters using traditional network models would not differ from the discretion of many pairwise interactions. Simplicial sets can also denote explicit sub-conversations and their relationship to one another through the use of face maps (a part of the formal definition of simplicial sets, see Appendix 1). Simplicial sets are generalizations of networks in that a network is itself a 1-dimensional simplicial set. Conversely, the *1-skeleton* of a particular simplicial set is the set of all 0-simplices (vertices) and 1-simplices (edges) and is a network by definition. Note however that because we lose information about the system when we move from a multi-dimensional simplicial set to the 1-skeleton (or network), it is not possible to recover the original simplicial set from the information contained in the 1-skeleton. This further implies that unique simplicial sets could reduce to the same 1-skeleton. Finally, as with any modeling approach, it is important to note that it is not always necessary to

choose the most complex tool when a simpler one will suffice.

3 Using Simplicial Models to Study Social Processes in Animal Groups

Communication of social animals can occur among multiple individuals simultaneously, within groups, between groups, and sometimes across species. The dynamics of such multi-agent communication are lost when using only traditional network approaches. Here we review various fields in animal behavior that may benefit from employing simplicial modeling approaches to better understand the causes and consequences of communication among social animals.

3.1 Signaling and communication

3.1.1 Alarm calls

Vervet monkeys use context-specific signals to warn conspecifics of impending danger. These calls not only indicate that danger is nearby, but also the type of danger, e.g. aerial predator vs. ground predator (Cheney and Seyfarth, 1992). The response of the entire group depends on the information contained in the warning signal. Furthermore, learning what signals to produce, and when to use them, depends on the type of signals individuals were exposed to during development (Seyfarth and Cheney, 1980; Seyfarth and Cheney, 1986). Simplicial sets can help uncover how such context specific signals are learned during the development of an individual. Because these alarm calls are broadcast, and not necessarily directed at particular individuals, modeling who learned which signal from whom using traditional network analysis will become cumbersome very quickly. Simplicial sets allow modeling the build-up of experience with a particular signal while taking into account simultaneous exposures to a signal at various times and considering the presence of different sets of individuals

from the larger group at each time point. Note that by strict definition, a simplicial set consists of ordered sets of simplices, which are themselves ordered sets of interacting vertices. This characteristic of simplicial sets allows us to easily designate a temporal ordering or even directionality (i.e., of information transfer) to the set of interactions. Bidirectional communication is implemented by adding simplices with the desired “opposite” directionality or by ignoring the defined ordering (similar to directed vs. undirected networks.)

3.1.2 Eavesdropping

Animals can benefit by listening to signals produced by other individuals even when they are not the intended receiver of the signal. For example, observing a fight among two competitors can provide valuable information for future encounters (Johnstone, 2001) and listening to predators can inform habitat choices (Emmering and Schmidt, 2011). Simplicial sets can depict this type of inadvertent information gathering more efficiently than traditional network analysis and can model the flow of such information among groups and over time.

3.1.3 Awareness Probing

Black-tailed prairie dogs *Cynomys ludovicianus* have been shown to actively probe the awareness and anti-predator vigilance of conspecifics (Hare et al., 2014). This probing involves a multi-modal display consisting of visual and auditory components causing a cascade of such displays through the population. Simplicial set models can cleanly implement both the temporal aspects of the cascade, as well as the spatial variance between the effects of the auditory and visual components of the display, and the feedback effects on the ratio of foragers to sentinels within the population.

3.2 Collective decision making

Recruitment to food in social insect colonies provides another example of communication via multi-way interactions among animals. For example, the use of pheromone trails provides information on the location of a food source to multiple workers simultaneously across large spatial and temporal scales (Traniello, 1989). On a smaller spatial and temporal scale, a honey bee forager provides information about the location and quality of a food source to multiple foragers simultaneously through an elaborate dance signal (Von Frisch, 1967). In both cases, information spreads via interactions, and collective knowledge of available resources emerges. Because the communication strength changes over time depending on reinforcement (i.e., how many workers lay the pheromone trail), we may want to represent communication as a function of the number of

participants. Therefore, to uncover how knowledge of available resources is shared within a colony, we need to understand how many workers are engaged in each activity and in which spatio-temporal pattern, because parallel signals may be produced by multiple groups. Simplicial sets allow us to examine how the number and placement of informed individuals, or the number of individuals influenced by a pheromone at any time, affects the spread of information throughout the colony and how the dynamics of this information spread influence the accumulation of global knowledge.

3.3 Fission fusion dynamics

Many social animals live in a fission-fusion society, i.e., individuals regularly change who they interact with over time (Aureli et al., 2008). For example, wild chimpanzee males interact in groups of varying sizes and composition to hunt, defend a territory, or guard a mate (Muller and Mitani, 2005). Each male may participate in several different groups, each with a different purpose, however, each male also seeks to assert its dominance over as many other males as possible. Thus, it is critical to coordinate a group’s collective responses in a way that promotes both one’s social dominance and its survival. Simplicial sets would allow us to easily represent multiple types of sub-group interactions (i.e., social grooming vs aggression) and the relationship among the various types of interactions. Furthermore, it will allow modeling the effect of various types of interactions on both local and global knowledge and on changes to the dominance hierarchy, and related interactions over time. Increased understanding of how the available information about dominance status and existing alliances percolates in this fission-fusion society, as well as the inherent bottlenecks that typify existing arrangements, could lead to a better evolutionary understanding of the social structure of this species.

3.4 Social learning

Studies of information sharing among animals often focus on who learns what from whom (Heyes and Galef, 1996). Social learning is most frequently defined as the acquisition of a previously undisplayed behavior via observation of a conspecific (Heyes, 1994), which has been demonstrated in several species (e.g., wild chimpanzees; Hobaiter et al., 2014). However, it is often difficult to control for environmental or genetic factors that might influence the acquisition of the observed behavior, and controlled experimental methods that examine dyadic interactions are often employed to determine whether particular behaviors are socially transmitted (Lonsdorf and Bonnie, 2010; Reader and Biro,

2010). More recently, network-based diffusion analysis (NBDA) has been used to examine information transmission in a more natural social setting of a group of interacting individuals (Franz and Nunn, 2009; Hoppitt et al., 2010). NBDA and similar methods offer statistical power to predict the existence of social learning in a group, given observations of behavioral adoptions over time, and can provide the potential pathway of information across an interaction network. However, these methods do not address the temporal change in the collective ability of the group as a whole to solve a task, nor do they allow for scenarios in which the dynamic of interactions in sub-groups is fundamentally different from that of pairwise interactions (see examples in Table 2).

Traditional networks are able to convey the fact that a particular agent had an influence of a particular magnitude on another agent, but this model breaks down when we wish to consider the combined effects of multiple signals and the effect of explicit subgroups on the dissemination of information (cf. Jeanne, 1996; Karsai and Wenzel, 2000; O'Donnell and Bulova, 2007; Robson and Traniello, 1999). We can obtain a better understanding of the knowledge capacity of social animal groups by using mathematical models that are capable of representing more than pairwise interactions. This will allow us to more accurately examine the cascades of information that occur when multiple animals observe a group member perform a novel behavior, or when information is conveyed by different subsets of the group at various times.

Next, we define and discuss a class of higher-order models as a concrete example of how to efficiently capture the dynamics of social transmission and knowledge capacity in animal groups.

4 Simplicial Sets in Application

Both human (Golub and Jackson, 2010; Kearns et al., 2006) and animal (Conradt and Roper, 2005; Couzin et al., 2005; Franks et al., 2002; Huse et al., 2002; Krueger et al., 2014) groups are capable of solving cognitively difficult problems better than single individuals. Interestingly, collective solutions can be reached efficiently when each group member has only limited access to information (Conradt and List, 2009; Kearns et al., 2006; Krause et al., 2010; Sasaki and Pratt, 2011; Sasaki and Pratt, 2012). Furthermore, information may be compiled via sub-group interactions to obtain solutions to problems (Kang and Xiong, 2013). Such information build-up can be limited by the communication structure itself,

i.e. who interacts with whom, as well as by the quantity or timing of interactions. Variability in individual capabilities to transmit information may play a role in how the social group accumulates and synthesizes knowledge (Pinter-Wollman et al., 2011) as will interactions between individuals with complementary sets of information (Garland et al., 2011). Finally, strategic sharing of information can play a large role in improving the knowledge capacity of a group (Sen et al., 1996). It is not always necessary for individuals to share their entire information set in each interaction – rather, creativity and innovation may be enhanced by sharing relevant results, without revealing all of the understanding that contributed to them. To understand how groups solve difficult problems that require information sharing we require methods that enable us to understand and describe multi-way interactions among sub-groups which traditional network methods cannot.

When attempting to analyze the build-up of knowledge from individual information to collective outcome within groups of communicating individuals, it is important to understand that communication among group members acts to coalesce individuals' information into a communal knowledge set that is potentially greater than the sum of the individual parts (Scardamalia and Bereiter, 2006). We call this collective knowledge set the 'knowledge capacity' of the group, and define the 'learning potential' of the group as the difference in its knowledge level between the final and initial distributions of knowledge. It is important to note our critical distinction between knowledge and information: We will refer to information when we speak of the singular things known by an individual – e.g. individuals A and B have information about the location of a food source; we refer to knowledge only when we speak of a set of information, whether that set is possessed by an individual, or built up through a group interaction – e.g., individuals A and B have interacted such that they now both possess identical knowledge about the location and quality of a certain food source. This distinction allows us to examine a range of individual capabilities with regard to sharing and acquiring information, as well as to understand the origins of creativity and innovation.

Our examination of knowledge capacity and learning potential differs from traditional research about information dissemination through social networks (Acemoglu et al., 2010; Bakshy et al., 2012; Bauch and Galvani, 2013; Boccaletti et al., 2006; Kempe et al., 2003; Singer, 2012), in that it allows the examination of the group inference processes. Percolation of informa-

tion is a dyadic process by which information is shared with some probability given an interaction between sender and receiver(s). By considering additional mechanisms through which the shared information is processed by the group, we transition from treating “knowledge” as a token that gets passed from teacher to learner to depicting knowledge more realistically as a set of building blocks from which new ideas may arise as an emergent property. Modeling knowledge capacity with simplicial models enables incorporation of functional responses based on the size of the interacting groups. Thus we can extend the insights provided by studies of information dissemination on networks to include the mechanisms by which knowledge building and innovation occur in social groups, especially with respect to the contribution of interactions among sub-groups.

An initial approach to examining the knowledge capacity of a group using simplicial sets begins by representing each member of the group by a 0-simplex, and each interaction between $k+1$ individuals as a k -simplex (generalizing the nodes and edges of a traditional network model). Formally, let the i^{th} individual in the target group be represented by a 0-simplex $X_i^0 \in X^0$. Let each k -simplex $X_j^k \in X^k$ be representative of an interaction between the $k+1$ individuals that comprise X_j^k . Then, given an initial knowledge value $b_{v_i}(0) \in \mathbb{R}^+$ for each vertex $v_i \in X^0$, the knowledge built within a particular simplex X_j^k at time t is given by

$$c(X_j^k, t) = \frac{1}{k+1} \sum_{v_i \in X_j^k} b_{v_i}(t)$$

We may now define a learning process, which is simply an updating rule for b_{v_i} at each time step, i.e. an accumulation of knowledge by all members of the simplex at that time step:

$$b_{v_i}(t+1) = \max\{b_{v_i}(t), \max c(X_j^k, t) \mid \forall X_j^k \in X^k \mid (v_i \in X_j^k)\}$$

With this approach, we have described learning within a particular communication interaction (k -simplex) in such a way that participating individuals with less knowledge than the group average have acquired new information that increases their knowledge to that of the group average of the most productive interaction in which they have participated. These dynamics are inspired by the DeGroot model of social learning (DeGroot, 2005) and chosen as a natural model for an initial exploration of knowledge and learning. A crucial

aspect of future research will be to find appropriate functions for each system of interest that accurately depict the accumulation of knowledge. For example, studies of social learning in animals may benefit from functions that capture how innovation and discovery of new knowledge emerges from social learning or communication. Further, note that by our definition of individual knowledge as a real-valued number, we have restricted our sample framework to a serial learning dynamic (i.e. an individual possessing a knowledge of “3” is understood to have the information needed to know “1” and “2”). This assumption may be relaxed in future work to enable the modeling of disjoint sets of individual information. Still, our current definition provides a simplified starting scenario for investigating the impact of higher-order topological structures on information sharing and knowledge building in social groups.

We now give a basic example of how such a framework could be utilized in application by providing the following scenario: Consider a group of 10 New Caledonian crows, a species that is known for its social learning and information transmission skills (Holzhaidner et al., 2010; Kenward et al., 2006). Each group member is categorized according to family relations: $\{v_0\}$ parent, $\{v_1, v_2, v_3\}$ offspring, and $\{v_4, \dots, v_9\}$ non-relatives. Suppose that information regarding tool design and use is shared first through pairwise meetings between parent and each individual offspring, as well as pairwise interactions between each pair of offspring. Subsequently, each offspring has a group interaction with a unique set of non-relatives, after which the offspring “report back” for a (simultaneous) group interaction with the parent. This scenario is encapsulated by the simplicial set geometrically represented in Fig. 5. Let each individual be represented by a vertex in $X^0 = \{v_0, \dots, v_9\}$. Then let each sharing interaction be given as $X^1 = \{(v_0, v_1), (v_0, v_2), (v_0, v_3), (v_1, v_2), (v_1, v_3), (v_2, v_3)\}$ $X^2 = \{(v_1, v_4, v_5), (v_2, v_6, v_7), (v_3, v_8, v_9)\}$ $X^3 = \{(v_0, v_1, v_2, v_3)\}$

To understand how the knowledge flows within this example scenario, we assign each individual a value representing the level of information they possess as shown in Table 1 ($t = 0$). Initial values of information, such as the ability to produce and use a particular tool, were given arbitrarily such that offspring begin with the smallest amount of information. We then assume that this information is shared within each interaction ac-

cording to equations 1 and 2. The sharing interactions detailed above can be captured by imposing a feed-forward ordering, such that time $t: 1 \rightarrow 3$. At each time step t , sharing interactions occur simultaneously within all k -simplices where $k = t$; i.e. at $t = 3$, we process the interaction on the simplex in X^3 . This was done not only as a simplifying assumption for this introductory example but as a way of placing a finite upper bound on the number of interactions in X ; more realistic communication protocols can be adopted according to the purpose of the model and empirical observations of the study system.

Table 1 Knowledge values at each time step t

	$t = 0$	$t = 1$	$t = 2$	$t = 3$
v_0	10	10	10	16.5
v_1	8	9	18	18
v_2	8	9	18.67	18.67
v_3	8	9	19.33	19.33
v_4	21	21	21	21
v_5	24	24	24	24
v_6	22	22	22	22
v_7	25	25	25	25
v_8	23	23	23	23
v_9	26	26	26	26

We give arbitrary initial knowledge values $b_{v_i}(0)$ for each vertex in the example of New Caledonian crows described in the text. Values at subsequent time steps are calculated using the learning equations 1 and 2.

By applying the learning equations 1 and 2 to the described communication structure, we can obtain final knowledge values for each vertex as shown in Table 1. If we consider the knowledge capacity as the average of the information possessed by each individual, then we obtain a final knowledge capacity B_{final} of 21.35, with a learning potential ΔB of 3.85. While we realize the triviality of characterizing the “state of all things known” in this way, we have supplied this definition of B for illustrative purposes only while making the case that more applicable functions can be defined according to the system of interest. Within this simple scenario, the influx of knowledge back to the parent may be of greater interest when studying social learning of behaviors, therefore Δb_{v_0} would be a more informative number than B_{final} .

While the example described above could have also been implemented using a hypergraph model (for definition and example, see Fig. 4), it is important to note

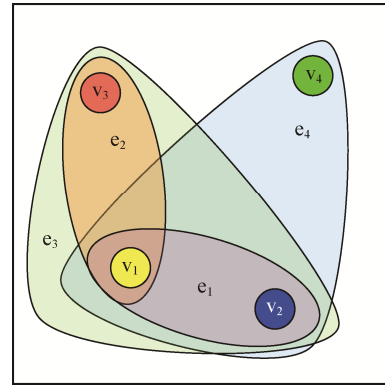


Fig. 4 An example hypergraph

A hypergraph connects vertices together using multi-edges, which enable us to connect more than two vertices together at a time. As shown, edges e_1 and e_2 constitute pairwise interactions. Edges e_3 and e_4 show interactions between 3 vertices. Note that within this framework, we could denote multiple distinct interactions between the same sets of vertices, although this is not depicted here. However, a problem arises when we ask a question such as “Is edge e_1 a sub-edge of e_3 or of e_4 ?” Because there is no notion of “sub-edge” within the definition of a hypergraph, this question cannot be accurately answered.

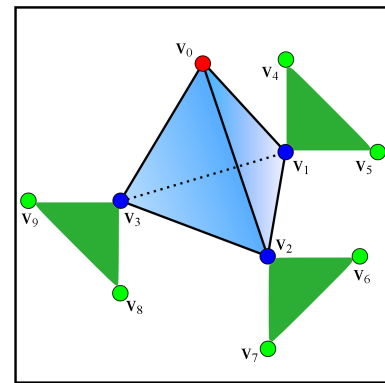


Fig. 5 Simplicial set representation of the new Caledonian crows example

A simplicial set depicting three sets of learning events among new Caledonian crows. In the first set, the parent (v_0 , in red) exhibits a learned task in front of its offspring (v_1, v_2, v_3 , in blue) as they interact with each other. These interactions are shown by black edges. In the second set of conversations, the offspring exhibit the learned behavior to unrelated individuals (in green) in groups of 3, shown by the green triangles. In the final set of interactions, the offspring return to the parent and interact as a group of 4, shown by the blue tetrahedron. See example calculations given in Table 1.

that the use of the learning equations we described above on simplicial architectures provide us with a flexible framework. For example, using our framework we can ask questions that cannot be answered using a hypergraph model without imposing significant additional assumptions and mathematical structure, such as:

How does the ordering of interactions influence the group’s learning potential? (e.g., is the learning poten-

tial greater if we reverse the ordering of the interactions $-t: 3 \rightarrow 1$ instead of $t: 1 \rightarrow 3$)

How does subgroup structure influence information flow? (e.g., what would happen to the learning potential if the offspring were to have a group sharing interaction before “reporting back” to the parent?)

How does the duration of, or number of participants in, interactions influence the speed vs. accuracy tradeoff that characterizes many collective decision making processes (McClurg, 2003)? (e.g., does the learning potential increase as duration or number of interactions increase, or is there an optimal number of interactions above which a collective decision no longer improves?)

Finally, we have provided (for comparative purposes) a more traditional network depiction of the example scenario (Fig. 6A), where an edge exists between any two vertices if the corresponding individuals interacted in any way, at any time. Note the fundamental difference in interpretation that occurs by attempting to model this scenario with a network, namely that we can observe *which* pairs of individuals have interacted to share information, but have no knowledge of whether any of these pairs were simultaneous interactions, or in which order the interactions might have occurred. Further, the limitation to only dyadic interactions captures different information about the system even if we were to take the extra step of invoking a time-ordering approach (Blonder and Dornhaus, 2011, Blonder et al., 2012, Pinter-Wollman et al., 2014) to the communications by invoking sub-graphs of G , over which communications

can occur at time t (Figs. 6B–D). Therefore, any network based method that we were to employ for calculations would overlook the functional effect of group size on learning that is critical to this example. This point serves to illustrate the importance of matching model framework to question and system.

5 Conclusions: Insights Gained through Higher-Order Analysis

Simplicial models have particular applicability to capturing scenarios in which it is important not only *that* individuals have interacted, but also *in what manner* they have done so; this opens up a litany of new avenues of exploration into efficient structures for communication. In this paper, we have given a variety of examples and applications to demonstrate the value of simplicial sets as a generalization of traditional network analysis. Furthermore, our framework allows investigating a broad spectrum of additional problem types that cannot be captured by networks such as increases in knowledge capacity and learning potential in social groups.

At this juncture, note that we do not propose simplicial models as a generic replacement for traditional network models, but rather assert their usefulness for questions that require the encapsulation of dynamics having higher dimensionality than simple pairwise interactions. We propose that these representations would be used to complement existing network techniques to better describe socially mediated transmission of information, thus enabling the field to ask deeper questions

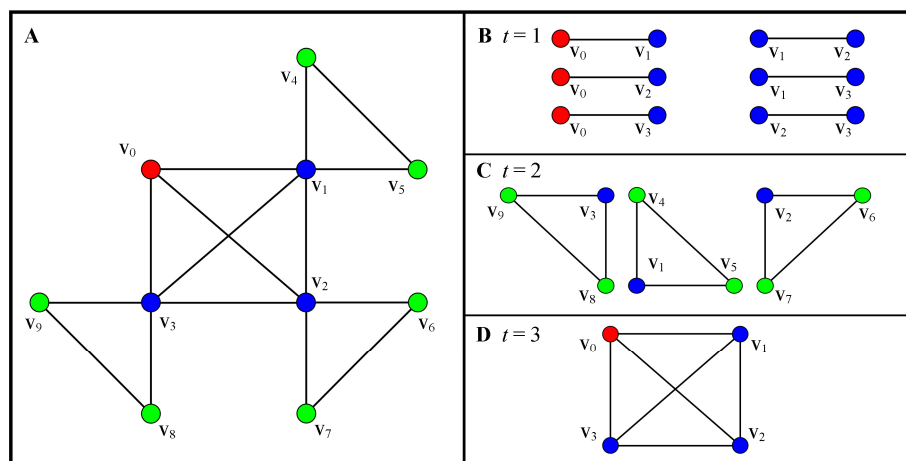


Fig. 6 Network representation of the new Caledonian crows example

A network representation of the set of interactions depicted in Figure 5. A) pairs of vertices are linked by an edge if they have participated together in any interaction. Note that this network shows accurately if two individuals have interacted, but falls short of representing the size of the conversation. B-D) the network in A broken into sub-graphs representing the interactions at each time t . The time-ordered approach may allow us to represent the conversation size, but does not faithfully represent the functional learning or sub-group participation dynamics described in the text. Further, it is easy to see that the sub-graph approach quickly becomes unwieldy as the set of vertices and interactions becomes large.

Table 2 Examples of Higher-Order Questions

Question	Applications
Two or more simultaneous flows; need to know which subset of individuals is associated with which flow, at what time, and to what effect	Simultaneous computer processes; campaigning strategies in political elections; nest-site selection in social insects; communication; comparative mate choice, parental behavior
Each subset of “receivers” of information is a separate entity and is associated with a (possibly) different cost	Optimal routing in multi-hop wireless networks; team formation; division of labor
The size of the interacting groups presents a critical difference in the interpretation of the interaction	Academic collaborations; team formation; semantic document clustering; optimal education strategies; comparative studies; group size effects
Interactions in explicit sub-groups have emergent properties which cascade temporally as well as across groups.	Origin of innovation via knowledge building in social groups; fomite-based disease transmission where infectious dose per individual is reduced by distribution as group size increases; vigilance networks

A sample of the types of questions for which higher-order modeling frameworks are useful and necessary, as well as broad applications for each question category.

about how communication structures mediate knowledge sharing and enable effective group decision making. A few such questions and applications are given in Table 2.

Simplicial sets provide a powerful set of analytical tools that move beyond their ability to generalize traditional network and hypergraph models, adding new ways to examine social processes. Thus, it is not surprising that simplicial models are applicable to a wide array of topics in both social and biological fields. For example, the problem of efficiently distributing information such that a country can choose a national political leader is fundamentally related to social insects disseminating information so that they can choose a new nest site. Strategies used for categorizing a set of search engine results (Do, 2012; Lin and Chiang, 2005) could be used to discover structure in the language of prairie dogs. Recent publications have used the ability of simplicial architectures to describe complex topologies to locate coverage holes in wireless networks (Ren et al., 2011), detect structured communities in large networks (Gopalan and Blei, 2013), and extract patterns from clouds of data points in *n*-dimensions (Chan et al., 2013; Nanda and Sazdanović, 2014). In addition to the uses illustrated here, we can see applications of this modeling class to biological topics such as protein interaction networks and quorum sensing in bacteria, as well as more human-centric topics such as optimal educational strategies and opinion propagation. By adopting and adapting these mathematical tools for use in biological systems, we hope to provide a foundation for multidisciplinary collaboration, identifying analogous problems across different domains, and providing an analytical framework for a unified exploration into these topics. The potential opportunities for comparing the efficiency of communication systems in social groups by analyzing knowledge capacity and learning potential with a

simplicial model are as varied as the communication systems they can represent.

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Appendix 1: Mathematical Definitions

Formally, let a *simplicial set* $X = \{X^0, \dots, X^n\}$ be a collection of simplices with maximum simplex dimension n , where X^k is the set of all simplices of dimension k , also referred to as the k -simplices of X . Then for each $k > 0$, let us define the following functions for each $0 \leq i \leq k$,

- $d_i: X^k \rightarrow X^{k-1}$; the *face maps* that take us from a particular k -simplex to the face of that k -simplex which does not contain its i -th vertex.
- $s_i: X^k \rightarrow X^{k+1}$; the *degeneracy maps* that take us from a particular k -simplex to the degenerate¹ $(k+1)$ -simplex which has the i -th element of the k -simplex repeated exactly twice.

These functions must obey the following identities (where $\circ$ denotes functional composition):

- $d_i \circ d_j = d_{j-1} \circ d_i$, if $i < j$
- $d_i \circ s_j = s_{j-1} \circ d_i$, if $i < j$
- $d_j \circ s_j = d_{j+1} \circ s_j$, identity
- $d_i \circ s_j = s_j \circ d_{i-1}$, if $i > j+1$
- $s_i \circ s_j = s_{j+1} \circ s_i$, if $i \leq j$

To make the above definition a bit easier to visualize, consider the example in Fig. 3 in which a (semi)-simplicial set $X = \{X^0, X^1, X^2\}$ is shown. Note that X is semi-simplicial because it does not feature any degenerate simplices, a choice made for ease of depiction. Then $X^0 = \{v_1, v_2, v_3, v_4\}$ represents the 0-simplices, or vertices, of the simplicial set. Similarly, $X^1 = \{(v_1, v_2), (v_1, v_3), (v_1, v_4), (v_2, v_4), (v_3, v_4)\}$ represents the set of 1-simplices, or edges, and $X^2 = \{(v_1, v_3, v_4)\}$ represents the lone 2-simplex, shown as a filled triangle. The face maps d_i point us to the face of a particular simplex that does not contain the i -th vertex of that simplex:

- 1-faces of the 2-simplex:
 - $d_0((v_1, v_3, v_4)) = (v_3, v_4)$
 - $d_1((v_1, v_3, v_4)) = (v_1, v_4)$
 - $d_2((v_1, v_3, v_4)) = (v_1, v_3)$
- 0-faces of the 1-simplices:
 - $d_0(v_1, v_4) = d_0(v_2, v_4) = d_0(v_3, v_4) = v_4$
 - $d_1(v_1, v_2) = d_1(v_1, v_3) = d_1(v_1, v_4) = v_1$
 - $d_0(v_1, v_3) = d_1(v_3, v_4) = v_3$
 - $d_0(v_1, v_2) = d_1(v_2, v_4) = v_2$

Finally, it is easy to check that our example constitutes a valid simplicial set by checking for set X^2 that the equality $d_i \circ (d_j) = d_{j-1} \circ (d_i)$ holds for the (i, j) combinations of (0,1), (0,2), and (1,2). Clearly, the combination of the sets of simplices together with the face maps is enough information for us to completely reconstruct the (semi)-simplicial set depicted in Fig. 3. If we had featured degenerate simplices in our example, the addition of the degeneracy maps would constitute the required information for reconstruction. Note that it is possible for a simplicial set to have non-degenerate simplices with degenerate faces and vice versa.

It is important to make our terminology explicit with respect to simplicial architectures, as there is some variation in the literature in this regard. We have given the definition of a *simplicial set* above, consistent with (Friedman, 2008). If

¹ A *degenerate simplex* is a simplex that has at least 1 repeated vertex and thus has a geometric realization of lesser dimensionality than its abstract representation. For example, in Fig. 3, if vertex v_1 and edge (v_1, v_2) existed twice, this would exemplify a 4-simplex that can be geometrically represented as a 3-simplex because the repeated vertex and edge exist in the same space.

we restrict the definition given of a simplicial set to disallow degenerate simplices, we have a *semi-simplicial set*. If we additionally restrict this definition such that each simplex in X is completely and uniquely identified by its set of vertices, we arrive at a *simplicial complex*. Rigorous definitions for simplicial complexes can be found in (Atkin, 1974; Friedman, 2008; Munkres, 1984; Spanier, 1994), while further definitions for semi-simplicial sets and simplicial sets can be found in (Spivak, 2009), where they are defined both in terms of their combinatorial definition as well as their category theoretic definitions.

We now address some terms and definitions that are important when using simplicial sets to capture group phenomena, followed by some sample applications. A k -simplex is a k -dimensional triangle constructed as the convex polyhedral hull (Berg et al., 2000) of $k+1$ vertices. For example, a 0-simplex is a vertex, a 1-simplex is an edge, a 2-simplex is a (filled) triangle, a 3-simplex is a (filled) tetrahedron, and so on. A *face* of a simplex is the polyhedral hull of a subset of the vertices in the simplex. Then, given these definitions, a *simplicial set* is the union of many such simplices, having the simplices “glued” together along shared faces. An n -dimensional simplicial set is a simplicial set such that n is the maximum size of all simplices in the set. If a particular simplex is not a proper subset of another simplex in a simplicial set X , then that simplex is referred to as a *facet* of X . The k -skeleton of a simplicial set X is the subset of X containing all simplices of at most dimension k .

Several familiar concepts from standard graph theory, and of frequent use in application to social network theory, sociology, and biology have been generalized to simplicial sets. We give the following generalizations of the familiar concepts of degree and adjacency, as detailed in (Muhammad and Egerstedt, 2006). The *upper degree* of a k -simplex $X_i^k \in X$ is the number of $(k+1)$ -simplices in X of which X_i^k is a face. Similarly, the *lower degree* of a k -simplex $X_i^k \in X$ is equal to the number of faces in X_i^k . Two k -simplices $X_i^k, X_j^k \in X$ are *upper adjacent* if both X_i^k and X_j^k are faces of a common $(k+1)$ -simplex in X . Likewise, two k -simplices $X_i^k, X_j^k \in X$ are *lower adjacent* if both X_i^k and X_j^k share a common face. Degree centrality, closeness centrality, and betweenness centrality (Freeman, 1979) have definitions generalized to simplicial complexes in (Jiang B and Omer I, 2007). Extending other structural characterizations (such as measures of clustering, c.f. (Bansal S, Khandelwal S and Meyers LA, 2009)) for use with simplicial topologies remain open areas of research.

It is easy to see an applied interpretation for measures such as these in various example scenarios. For example, suppose we wish to analyze the informational relationship among a group of research papers. We could construct a simplicial set in which each separate paper is represented by a vertex with a simplex existing only if all research papers that exist in that simplex share a topical keyword. In such a model, upper adjacency of two simplices (as defined above) could indicate that two topical keywords are (strictly) sub-topics of a broader, more encompassing keyword (e.g., “graph theory” and “algebraic topology” as sub-topics of “mathematics”). Similarly lower adjacency of two simplices could show that particular set of papers as having a mutual “inter-disciplinary” focus, since they share a subset of topical keywords that do not necessarily share the same broader topic. Note that two simplices may be both upper adjacent and lower adjacent to each other.

For another example, imagine a simplicial model of a scientific collaboration database in which each author is represented by a vertex, with each k -simplex of vertices (collection of $(k+1)$ authors) representing a joint publication. We may wish to analyze the relative importance of each author or collaborative effort to the field. In this case we could use the extensions of centrality measures as given in (Jiang and Omer, 2007) to compute the degree, closeness, or betweenness of each author or collaborative group. Examination of the topological holes and the *minimal non-faces* (cf. (Moore et al., 2012)) of the simplicial set give us two separate ways to identify missed collaborations in the group. Note as well that the 1-skeleton of such a simplicial set would represent the underlying traditional collaboration network notating simply which pairs of individuals had co-authored with each other.

The use of simplicial sets to model group processes can also, in some cases, allow for identification of new structural features within a group such as topological holes. Topological holes are breaks in the connectivity of the topological space (i.e. the donut hole in the center of a torus); with respect to communicative interactions, they may denote possible bottlenecks to information dissemination. Because simplicial sets can be analyzed algebraically (May, 1992), categorically (Grandis, 2002; Spivak, 2009), and combinatorially (Goerss and Jardine, 2009; Kozlov, 2008), they provide a versatile toolkit by which to examine new and mathematically interesting problems.